10. KdV equation and Solitary wave

10.1 Finite Amplitude Long Wave

Thus far, we have neglected the non-linear effect of waves. However, the height of tsunami waves is not satisfactorily small as compared with the depth and wavelength. We introduce two types of "small parameters" here: one is the ratio of the wave height a to depth D, $\varepsilon \equiv a/D$, and the other is the ratio of depth D to the wavelength L, $\delta = D/L$. ε denotes non-linearity while δ represents a long wave parameter. They are independent of each other. Based on these parameters, we can classify the waves into four categories:

(1) Both ε and δ are small: For example, long wave $c = \sqrt{gD}$ (10.1)

(2) ε is small, and δ is not always small:

Examples include linear waves (Airy's infinite amplitude wave), deep waves, and shallow water waves.

$$c = \sqrt{\frac{gL}{2\pi}\tanh\frac{2\pi D}{L}},$$

for the case of $D \to \infty$: Deep water wave $c = \sqrt{\frac{gL}{2\pi}}$

(10.2),(10.2')

(3) δ is small, and ε is not always small:

Long wave of finite amplitude, which is solvable by the characteristic curve method. Examples are bore and shock waves

(4) Both ε and δ are not always small:

Examples

Stokes wave to Levi Civita's waves

Trocoidal wave (Gerstner's wave), a waves that can be solved by the Lagrangian Method,

In this chapter, we will solve for the case that ε and δ are small but not infinitely small (category (3)).

Ursell's assumption

In the present study, we assume Ursell's assumption:

$$O(\delta) = O(\varepsilon^{1/2}) \quad (10.3)$$

and Ursell's parameter

$$Ur = \varepsilon / \delta^2 = aL^2 / D^3 \qquad (10.4)$$

is a moderate number

10.2 Expansion of the Velocity Potential Function in Taylor series

We assume that the motion of sea water is a non-vortex and that the velocity potential function ϕ exists; further, the flow (u, v) is given by the following:

$$(u, v) = (-\partial\phi / \partial x, -\partial\phi / \partial y) \qquad (10\text{-}5)$$

We consider the case that the sea depth is uniform $(= D)$, and we assume the x-axis in the direction of the wave, and y -axis perpendicular to it. The origin is set at a point on the undisturbed sea surface.

The equation of motion (Mass conservation equation) $(\partial u / \partial x + \partial v / \partial y = 0)$ is expressed by using the velocity potential function ϕ in the following form

$$\nabla^2 \phi = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = 0 \qquad (10\text{-}6)$$

The sea bed condition is given by

$$v = 0 \quad \text{on} \quad y = -D \qquad (10\text{-}7)$$

The velocity potential function can be generally expressed in the form $\phi = \phi(x, y, t)$ and is also expressed in the form of the Taylor series of y around the sea bed $y = -D$

$$\phi = \phi_0 + \phi_1 (y + D)^1 + \phi_2 (y + D)^2 + \phi_3 (y + D)^3 + \cdots \qquad (10\text{-}8)$$

Here $\phi_0, \phi_1, \phi_2, \phi_3 \cdots$ are subordinate functions of x, t alone.
(10-7) can be written by using φ:

$$\left[\frac{\partial \phi}{\partial y} \right]_{y=-D} = \phi_1 = 0 \qquad (10\text{-}9)$$

By substituting (10-8) in (10-6), we have (suffix “x” means differentiation of x)

$$\sum_{n=0}^{\infty} \left\{ \phi_{n,xx} (y + D)^n + n(n-1)\phi_n (y + D)^{n-2} \right\} = 0$$

In other words,

$$\sum_{n=0}^{\infty} \left\{ \phi_{n,xx} + (n+1)(n+2)\phi_n \right\} (y + D)^n = 0 \qquad (10\text{-}10)$$

(10-10) should be satisfied for any x, y in such a manner that all the terms in $\{ \}$ are zero.We then have

$$\phi_k = \frac{1}{k(k-1)}\phi_{k-2}, \qquad k = 2,3,\cdots \qquad (10\text{-}11)$$

When k is an odd number, i. e., $k = 2m+1$,

$$\phi_{2m+1} =$$

$$-\frac{1}{(2m+1)2m}\phi_{2m-1} = \frac{1}{(2m+1)2m(2m-1)(2m-2)}\phi_{2m-3} = \cdots = (-1)^m \frac{1}{(2m+1)!}\left(\frac{\partial}{\partial x}\right)^{2m}\phi_1 = 0$$

Thus, the even number terms vanished.

[Question 1] In general, even if $f(x)=0$ at $x=a$, $f'(x)$ is not always zero at $x=a$. Then why can we say that $f'(a)=0$ in this discussion ?

When k is an even number, i. e, $k = 2m$,

$$\phi_{2m} = (-1)^m \frac{1}{(2m)!}\phi_0 \qquad (10\text{-}10)$$

Equation (10-6) finally becomes

$$\phi = \phi_0 - \frac{(y+D)^2}{2!}\phi_{0.xx} + \frac{(y+D)^4}{4!}\phi_{0,xxxx} - \cdots \qquad (10\text{-}13)$$

［Question 2］ Prove that this equation can be expressed in the following form

$$\phi = \cos\left\{(y+D)\frac{\partial}{\partial x}\right\}\phi_0 \qquad (10\text{-}14)$$

10.3 Order Estimation

We assume that the order of the magnitude of the first term of (10-13) is unity. The next term $-(y+D)^2/2!\times(\partial^2\phi_0/\partial x^2)$ is obtained by the following procedure.

φ is multiplied by $(y+D)^2$ and is differentiated twice with respect to x. Thus, its order is estimated as φ × D^2/L^2, and hence, the second term has the order of

$$\varphi \times O(D^2/L^2) = \delta^2 = \varepsilon^1 \varphi$$

Similarly, the third term of (10-13) has the order of ε^2 φ.

10.4 Kinematic sea surface condition

The kinematic sea surface condition in the strict form is given by

$$v = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} \qquad \text{with} \quad y = \eta(x,t) \qquad (10\text{-}15)$$

(10-15) is satisfied if $y = \eta$, that is,

$$[v]_{y=\eta} = \left[-\frac{\partial\phi}{\partial y}\right]_{y=\eta}$$

We transfer this into an equation satisfied when $y=0$. By applying the Taylor series for y=0, we have

$$\left[-\frac{\partial\phi}{\partial y}\right]_{y=\eta} = \left[-\frac{\partial\phi}{\partial y}\right]_{y=0} + \left[-\frac{\partial^2\phi}{\partial y^2}\eta\right]_{y=0} + \left[-\frac{1}{2!}\frac{\partial^3\phi}{\partial y^3}\eta^2\right]_{y=0} + \cdots \qquad (10\text{-}16)$$

Order→ 1 ε^1 ε^2

We only select the highest and second highest terms; the kinematic sea surface condition (10-13) then finally takes the following form:

$$\left[-\frac{\partial\phi}{\partial y}\right]_{y=0} + \left[-\frac{\partial^2\phi}{\partial y^2}\right]_{y=0} = \frac{\partial\eta}{\partial t} + \left[-\frac{\partial\phi}{\partial x}\right]_{y=0}\frac{\partial\eta}{\partial x} \qquad (10\text{-}17)$$

Order → 1 ε 1 ε

［Question 3］ No water is present when $y=0$ at the trough of the wave, and hence, we cannot determine the “current velocity” (u, v) at such a location. How should we interpretation such a point keeping (10-17) in mind?

10.5 Dynamic sea surface condition

Dynamic (pressure) sea surface condition is given by Bernoulli’s Equation as follows:

$$\left[-\frac{\partial\phi}{\partial t}\right]_{y=\eta} + g\eta + \frac{1}{2}\left(u^2+v^2\right) = const. \quad \text{for } y=\eta \qquad (10.18)$$

Let us try to estimate the orders of these four terms by using the following order estimation table.

$$O(g) = c^2/D = O(L^2/(DT^2)) = O(\delta^{-2}D/T^2) = O(\varepsilon^{-1}D/T^2)$$
$$O(g\eta) = O(\varepsilon^{-1}D/T^2\times a) = O(\varepsilon^{-1}D/T\times\varepsilon^1 D) = O(D^2/T^2)$$
$$O(v) = O(a/T) = O(\varepsilon^1 D/T), O(v^2) = O(\varepsilon^2 D^2/T^2)$$
$$O(u) = O(v\times L/D) = O(\varepsilon^1 D/T\times L/D) = O(\varepsilon^1\delta^{-1}D/T)$$
$$O(u^2) = O(\varepsilon^2\delta^{-2}D^2/T^2) = O(\varepsilon^1 D^2/T^2)$$

(10.19-a,b,c,d,e,f)

The order of the first term should be balanced with the second term, and hence,

$$O(\left[-\frac{\partial\phi}{\partial t}\right]_{y=\eta}) = O(g\eta) = O(D^2/T^2) \qquad \text{(10.19-g)}$$

Thus we choose the terms up to the order of

$$\left[-\frac{\partial\phi}{\partial t}\right]_{y=\eta} + g\eta + \frac{1}{2}u^2 = const. \quad \text{for } y = \eta \qquad \text{(10-20)}$$

We again apply the Taylor series expansion around $y = 0$ for the first term, thereby obtaining

$$\left[-\frac{\partial\phi}{\partial t}\right]_{y=\eta} = \left[-\frac{\partial\phi}{\partial t}\right]_{y=0} + \left[-\frac{\partial^2\phi}{\partial y\partial t}\right]_{y=0}\eta \qquad \text{(10-21)}$$

After estimating the order of the second term, we find that

$$O\left(\left[-\frac{\partial^2\phi}{\partial y\partial t}\right]_{y=0}\eta\right) = O(\varepsilon^1 D^2/T^2) \qquad \text{(10.22)}$$

Thus we have the final form of the dynamic boundary condition on the sea surface as follows:

$$\left[-\frac{\partial\phi}{\partial t}\right]_{y=0} + \left[-\frac{\partial^2\phi}{\partial y\partial t}\right]_{y=0}\eta + g\eta + \frac{1}{2}\left[u^2\right]_{y=0} = const. \qquad \text{(10.23)}$$

Order→ 1 ε^1 1 ε^1

Thus, the basic equations in the present problem to be solved are (10-17) and (10-23), and the unknowns are φ and η.

10.6 Zero-th order solution

We only select the maximum order terms in (10-17).

$$\left[-\frac{\partial\phi}{\partial y}\right]_{y=0} = \frac{\partial\eta}{\partial t} \qquad \text{(10-17-a)}$$

Substituting the second term of (10-13) in (10-17-a) gives

$$\left[-\frac{\partial\phi}{\partial y}\right]_{y=0} = \left[(y+D)\phi_{0,xx}\right]_{y=0} = D\phi_{0,xx}$$

Hence, we have

$$-D\phi_{0,xx}+\frac{\partial \eta}{\partial t}=0 \qquad (10\text{-}24)$$

On the other hand, we choose the maximum order term, and then we have

$$-\frac{\partial \phi_0}{\partial t}+g\eta=0 \qquad (10\text{-}25)$$

Eliminating ϕ_0, we obtain

$$\frac{\partial^2 \eta}{\partial t^2}=c_0^2\frac{\partial^2 \eta}{\partial x^2} \qquad (10\text{-}26)$$

This is a general form of the equation of a wave, and has the following solution:

$$\eta=f_+(x-c_0t)+f_-(x+ct) \qquad (10\text{-}27)$$

Here, f_+, f_- are the wave components in the positive and negative x directions, respectively.

Note that $\eta=f_+(x-ct)$ satisfies the following relationship:

$$\frac{\partial \eta}{\partial t}=-c_0\frac{\partial \iota}{\partial x} \qquad (10\text{-}28)$$

Hereafter, we only consider the wave components moving in the positive x direction.
Substituting (10-28) in (10-25) yields

$$D\phi_{0,xx}+c_0\frac{\partial \eta}{\partial x}=0 \qquad (10\text{-}29)$$

Integrating wrt x leads to

$$u_0=\frac{c_0}{D}\eta \qquad (10\text{-}30)$$

10.7 **First-Order Solution**

We now proceed to the first-order solution: We choose all the terms in (10-17).
By using (10-30) and the third term of (10-16) into the last term of (10-17), we obtain

$$g\eta-\frac{\partial \phi_0}{\partial t}+\frac{c_0^2}{2}(D\eta_{xx}+\frac{1}{D^2}\eta^2)=0 \qquad (10\text{-}31)$$

On the other hand, the dynamic sea surface condition (10-23) is reduced similarly, yielding

$$\eta_t-D\phi_{0,xx}+c_0\frac{\partial}{\partial x}(\frac{1}{D}\eta^2-\frac{D^2}{6}\eta_{xx})=0 \qquad (10\text{-}32)$$

By eliminating ϕ_0 from (10-31) and (10-32), we have

$$\frac{\partial^2 \eta}{\partial t^2}=c_0^2\frac{\partial^2 \eta}{\partial x^2}+c_0^2D\frac{\partial^2}{\partial x^2}\left(\frac{3}{2}\frac{1}{D^2}\eta^2+\frac{1}{3}D\eta_{xx}\right) \qquad (10\text{-}33)$$

By comparing this equation with (10-26), we find that the term of () is added to it, and this is a first-order small term.

We set the first-order term to be $P(\eta)$, that is,

$$P(\eta)\equiv D\left(\frac{3}{2}\frac{1}{D^2}\eta^2+\frac{1}{3}\eta_{xx}\right) \tag{10-34}$$

Then, since $P(\eta)$ has only one subordinate valuable η, we have the first order approximation

$$\frac{\partial P}{\partial t}=-c_0\frac{\partial P}{\partial x} \tag{10-35}$$

（Note） $\eta\approx\eta(x-ct)$, and hence, P also satisfies $P\approx P(x-ct)$, and (10-35) is satisfied.

(10-51) can be expressed in the form

$$\frac{\partial^2\eta}{\partial t}=c_0^2\frac{\partial^2}{\partial x^2}(\eta+P) \tag{10-36}$$

We consider $F(\eta)$ to be a second-order unknown function satisfying

$$\eta_t=-c_0\frac{\partial}{\partial x}\{\eta+F(\eta)\} \tag{10-37}$$

We substitute this into (10-36), and it takes the following form

$$-c_0\frac{\partial}{\partial x}\frac{\partial}{\partial t}(\eta+F)=c_0^2\frac{\partial^2}{\partial x^2}(\eta+P)$$

Dividing by c_0 and integrating with respect to x, we have

$$-\frac{\partial}{\partial t}(\eta+F)=c_0\frac{\partial}{\partial x}(\eta+P)$$

We again substitute this into (10-55), giving

$$c_0\frac{\partial}{\partial x}(\eta+F)-\frac{\partial F}{\partial t}=c_0\frac{\partial\eta}{\partial x}+c_0\frac{\partial P}{\partial x}$$

$$\therefore\quad c_0\frac{\partial F}{\partial x}-\frac{\partial F}{\partial t}=c_0\frac{\partial P}{\partial x} \tag{10-38}$$

We rewrite the right-hand side of this equation into two parts as follows:

$$c_0\frac{\partial P}{\partial x}=\frac{1}{2}c_0\frac{\partial P}{\partial x}+\frac{1}{2}c_0\frac{\partial P}{\partial x}\cong\frac{1}{2}c_0\frac{\partial P}{\partial x}-\frac{1}{2}\frac{\partial P}{\partial t}$$

By comparing this equation and the left-hand side of (10-38), we find out that the unknown function $F(\eta)$ can be set as $F(\eta)=1/2P(\eta)$.

Thus, we transfer (10-51) into the following form in order to accommodate the positive x component as follows:

$$\eta_t+c_0\eta_x+c_0D\frac{\partial}{\partial x}\left(\frac{3}{4}\frac{1}{D^2}\eta^2+\frac{1}{6}D\eta_{xx}\right)=0 \tag{10-39}$$

We call this equation as the Korteweg-de-Vries (KdV) Equation for an ocean wave.

10.8 Normal form of KdV equation

We introduce the transfer of the independent valuables in (10-39) as follows:

$$X = x - c_0 t, T = t \qquad (10\text{-}40)$$

and the subordinate valuable η as

$$\xi = \frac{3}{2}\frac{c_0}{D}\eta,$$

The coefficient is set as

$$\beta \equiv \frac{c_0}{6} D^2$$

Then we finally obtain the normal form of KdV Equation as

$$\xi_T + \xi\xi_X + \beta\xi_{XXX} = 0 \qquad (10\text{-}41)$$

Moreover, if we introduce the following transformation

$z = \frac{X}{\sqrt{\beta}}, \tau = \frac{T}{\sqrt{\beta}}$ Then (10-41) becomes

$$\xi_\tau + \xi\xi_z + \xi_{zzz} = 0 \qquad (10\text{-}42)$$

10.9 Soliton Solution of the KdV Equation

Let us solve (10-41) by assuming that the shape of the wave does not changes permanently. The KdV equation (10-41) has a solution of a permanent type, with only one peek.

We assume that an observer moves on a car with a speed γ. (10-42) then has a permanent solution $\xi = \xi(X - \gamma T)$ that satisfies

$\frac{\partial \xi}{\partial T} = -\gamma \frac{\partial \xi}{\partial X}$ (20-41) takes the following form

$$-\gamma\xi_X + \xi\xi_X + \beta\xi_{XXX} = 0 \qquad (20\text{-}43)$$

It is possible to integrate this wrt X in the limit between $\{-\infty, X\}$; we give the condition at the left infinitive as $[\xi]_{X=-\infty} = 0$, thereby obtaining

$$-\gamma\xi + \frac{1}{2}\xi^2 + \beta\xi_{XX} = 0 \qquad (10\text{-}44)$$

We multiply this by $2\xi_X$, it wrt X, and the result is as follows:

$$-\gamma\xi^2 + \frac{1}{3}\xi^3 + \beta\xi_X^2 = 0$$

This equation can be solved by ξ_X as

$$\xi_X = \xi\sqrt{\frac{\gamma}{\beta} - \frac{1}{3\beta}\xi} \qquad (10\text{-}45)$$

This is a differential equation of a variables separable type, and it is easily solved. We simply use

$-\frac{1}{3\beta} \equiv a, \frac{\gamma}{\beta} \equiv b$ (10-63) becomes

$$X = \int \frac{d\xi}{\xi\sqrt{a\xi + b}} \qquad (10\text{-}45b)$$

This is solved in the following form, in which we set the integral constant C as γT because the final solution should be the form $\xi = f(X - \gamma T)$.

$$X = \frac{1}{\sqrt{b}}\log\left|\frac{\sqrt{a\xi + b} - \sqrt{b}}{\sqrt{a\xi + b} + \sqrt{b}}\right| + \gamma T \qquad (10\text{-}46)$$

(See “Sugaku Koshiki 1” by “Iwanami press”, p95”

This equation can be solved by ξ in the following form

$$\xi = -\frac{b}{a}\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{b}(X - \gamma T)\right) \qquad (10\text{-}47)$$

［Question］ Explain why we make the integral constant C as γT in calculation of (10-47)

［Question］ Derive (10-65) from (10-64).

We re-set the parameters using the original style, yielding

$$\xi = 3\gamma \operatorname{sec}h^2\left\{\frac{1}{2}\sqrt{\frac{\gamma}{\beta}}(X - \gamma T)\right\} \qquad (10\text{-}48)$$

We put $3\gamma \equiv A$, and we have the formula of the form of a solitary wave as follows:

$$\xi = A\operatorname{sec}h^2\left\{\frac{1}{2}\sqrt{\frac{A}{3\beta}}(X - \gamma T)\right\} \qquad (10\text{-}49).$$

This is the solution of the normal form of the KdV equation (10-41). (10-49) shows a curve of one symmetry peek, similar as the normal distribution curve in statistics.

A is the height of the peek and is only one control parameter of the solution; in other words, once the wave height A is decided, the “effective length” L is also decided at the same time. We again re-set the original variables in (10-49), and we then have the following dimensional form as

$$\eta = \frac{2}{3}\frac{AD}{c_0}\sec h^2\left\{\frac{1}{2}\sqrt{\frac{A}{3\beta}}(x-ct)\right\} \qquad (10\text{-}50)$$

where $c = c_0 + A/3$. We introduce the real (actually visual) waveform by introducing a real wave height $H(=\frac{2}{3}\frac{AD}{c_0})$ We have the final form as

$$\eta = H\sec h^2\frac{1}{2}\sqrt{\frac{3H}{D^3}}(x-ct) \qquad (10\text{-}51)$$

and

$$c = c_0\left(1+\frac{H}{2D}\right) \qquad (10\text{-}52)$$

Note that the local depth at the peek is $D+H$ and the velocity of the long wave is given by

$$c = \sqrt{g(D+H)} = \sqrt{gD}\left(1+\frac{H}{D}\right)^{\frac{1}{2}} \approx c_0\left(1+\frac{H}{2D}\right) \qquad (10\text{-}53)$$

This is very similar to the velocity formula (10-52).